

## Dot Product (Inner or Scalar Product)

Let  $\vec{a}, \vec{b} \in \mathbb{R}^n$  &  $\vec{a} = \langle a_1, a_2, a_3 \rangle$   $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \cdot \vec{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Note: You can only dot 2 vectors of the same  $\mathbb{R}^n$

Note: Dot product takes 2 vectors and spits out a scalar

Thm. Properties of Dot Products (Proofs on last page)

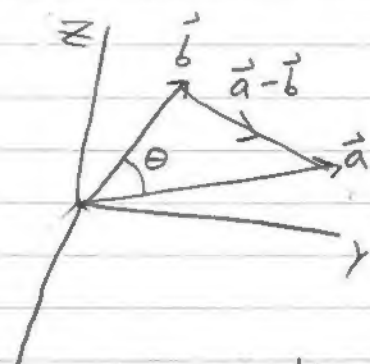
Let  $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$  and  $d \in \mathbb{R}$

- ① 0 Property:  $\vec{0} \cdot \vec{a} = 0$
- ② Magnitude Property:  $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$
- ③ Commutative Property:  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- ④ Constant multiple Property:  $(d\vec{a}) \cdot \vec{b} = d(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (d\vec{b})$
- ⑤ Distributive Property:  $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

Note:  $\vec{a} \cdot (\vec{b} \cdot \vec{c}) = (\vec{a} \cdot \vec{b}) \cdot \vec{c}$  is meaningless b/c when 2 vectors are dotted they produce a scalar, and you can only dot a vector with a vector, so which ever dot operation is 2<sup>nd</sup> will be a meaningless operation

Thm.  $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$  where  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$

Pf: Let  $\vec{a}, \vec{b} \in \mathbb{R}^3$



Since the 3 vectors form a triangle the Law of cosines applies

$$\|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos(\theta)$$

x Focus on the left side, apply Magnitude Property

$$\|\vec{a} - \vec{b}\|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$$

$$= (\vec{a} - \vec{b}) \cdot \vec{a} - (\vec{a} - \vec{b}) \cdot \vec{b}, \text{ Distributive Property} \quad (2)$$

$$= \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}, \text{ Distributive Property}$$

$$= \|\vec{a}\|^2 - 2\vec{a} \cdot \vec{b} + \|\vec{b}\|^2, \text{ Commutative and Magnitude Properties}$$

Bringing back the equality,  $\|\vec{a}\|^2 - 2\vec{a} \cdot \vec{b} + \|\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos(\theta)$

$$\text{Simplifying, } \vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$$

$$\text{Corollary: } \theta = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}\right)$$

$$\text{Corollary: If } \vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \text{ then } \vec{a} \parallel \vec{b}$$

$$\text{Corollary: If } \vec{a} \cdot \vec{b} = -\|\vec{a}\| \|\vec{b}\| \text{ then } \vec{a} \text{ is anti-parallel to } \vec{b}$$

$$\text{Corollary: If } \vec{a} \cdot \vec{b} = 0 \text{ } \vec{a} \perp \vec{b} \text{ (orthogonal)}$$

Direction & cos's, Given  $\vec{a} \in \mathbb{R}^3$

Direction & of non-zero vector  $\vec{a}$  are the &  $\alpha, \beta, \gamma$  in  $I = [0, \pi]$  that  $\vec{a}$  makes with the positive x, y, and z axes

Direction cos's are the cos's of  $\alpha, \beta, \gamma$

Using the Dot Product with  $\hat{i}, \hat{j}, \hat{k}$  we can find cos of an &

$$\cos(\alpha) = \frac{\vec{a} \cdot \hat{i}}{\|\vec{a}\| \|\hat{i}\|} = \frac{a_1}{\|\vec{a}\|} \quad \text{Similar logic } \cos(\beta) = \frac{a_2}{\|\vec{a}\|} \quad \cos(\gamma) = \frac{a_3}{\|\vec{a}\|}$$

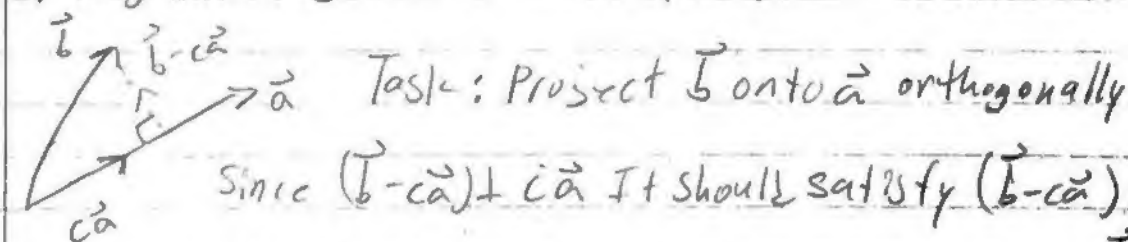
$$\text{Corollary: } \vec{a} = \langle a_1, a_2, a_3 \rangle = \langle \|\vec{a}\| \cos(\alpha), \|\vec{a}\| \cos(\beta), \|\vec{a}\| \cos(\gamma) \rangle$$

$$\vec{a} = \|\vec{a}\| \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle \rightarrow \frac{\vec{a}}{\|\vec{a}\|} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$$

Corollary: Square each expression for cos of an & and add

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = \frac{a_1^2 + a_2^2 + a_3^2}{\|\vec{a}\|^2} = \frac{(\sqrt{a_1^2 + a_2^2 + a_3^2})^2}{\|\vec{a}\|^2} = \frac{\|\vec{a}\|^2}{\|\vec{a}\|^2} = 1$$

Orthogonal Projections Let  $\vec{a}, \vec{b} \in \mathbb{R}^n$  and  $c \in \mathbb{R}$



Task: Project  $\vec{b}$  onto  $\vec{a}$  orthogonally

Since  $(\vec{b} - c\vec{a}) \perp c\vec{a}$  It should satisfy  $(\vec{b} - c\vec{a}) \cdot c\vec{a} = 0$

$$(\vec{b} - c\vec{a}) \cdot c\vec{a} = 0 \quad \vec{b} \cdot c\vec{a} - c\vec{a} \cdot c\vec{a} = 0 \quad \& (\vec{b} \cdot \vec{a}) = c(\vec{a} \cdot \vec{a}) \quad c = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2}$$

It follows that the projection of  $\vec{b}$  on  $\vec{a} = \text{proj}_{\vec{a}} \vec{b} = c\vec{a} = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$ .  
This is a scaled down version of  $\vec{a}$

Also  $\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|} \frac{\vec{a}}{\|\vec{a}\|} = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|} \left( \frac{\vec{a}}{\|\vec{a}\|} \right)$ . So  $\text{Comp}_{\vec{a}} \vec{b} = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|}$  which is the

Scaled unit vector in the direction of  $\vec{a}$  that gives  $c\vec{a}$

# Proof of Properties Dot Product

①  $\vec{a} \cdot \vec{0} = 0$   $\langle a_1, a_2, a_3 \rangle \cdot \langle 0, 0, 0 \rangle = a_1(0) + a_2(0) + a_3(0) = 0$

②  $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

$$\langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1^2 + a_2^2 + a_3^2 \geq 0 \rightarrow$$

$$(\sqrt{a_1^2 + a_2^2 + a_3^2})^2 = \|\vec{a}\|^2$$

③  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

$a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3$  Multiplication is commutative  $\rightarrow a_1b_1 + a_2b_2 + a_3b_3 = a_1b_1 + a_2b_2 + a_3b_3$

④  $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$

$$\langle ca_1, ca_2, ca_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = c \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle \\ = \langle a_1, a_2, a_3 \rangle \cdot \langle cb_1, cb_2, cb_3 \rangle$$

$$ca_1b_1 + ca_2b_2 + ca_3b_3 = ca_1b_1 + ca_2b_2 + ca_3b_3 = c(a_1b_1 + a_2b_2 + a_3b_3)$$

⑤  $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

$$\langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle = a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3)$$

$$a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) = a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1 + a_2c_2 + a_3c_3$$

$$a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 = a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1 + a_2c_2 + a_3c_3$$